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# FREQUENCY ASYMPTOTES AND ASYMPTOTIC ERROR FOR DISTRIBUTED-PARAMETER SYSTEMS

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# FREQUENCY ASYMPTOTES AND ASYMPTOTIC ERROR FOR DISTRIBUTED-PARAMETER SYSTEMS

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## SUMMARY

The frequency response techniques for lumped parameter systems are reviewed and extended to distributed-parameter circuit elements. For example, a three-terminal RC network can be specified by a set of asymptotes for amplitude and phase with an error usually less than 2 db or 20 degrees respectively. For any two-port a preferred set of six asymptotes can be defined from which asymptotes for all network function can be derived. This set of functions, referred to as deviations, corresponds to network polynomials in lumped parameter circuits. Properties of deviations are summarized and adapted for computation of frequency asymptotes and asymptotic error estimates of distributed-parameter systems. Illustrative examples include uniform and tapered thin-film networks.

## INTRODUCTION - SCOPE OF INVESTIGATION

### Objectives

Conventional circuit design concepts and techniques must be extended and amended, as distributed parameter networks are replacing lumped parameter networks, in an increasing range of circuit applications. Film-type networks not only meet the requirements for greater reliability, improved producibility and reproducibility, as well as lower cost and often smaller physical structures, but in addition have distinct advantages in circuit characteristics. For example, distributed R-C structures can realize sharper cut-off low-pass filters than lumped R-C circuits.

As part of design procedures, a circuit designer needs analytical expressions representing network-functions for different circuit configurations. Distributed parameter networks, such as transmission lines, lead to network characteristics described by transcendental functions in the complex frequency plane. Appropriate approximations of these transcendental functions provide sufficiently accurate design information for thin-film circuits. This investigation is directed to investigate the frequency response of distributed-parameter systems by developing:

- (1) An analytical approach to asymptotic approximations
- (2) Design procedures useful for film-type systems
- (3) Universal reference charts or curves for response characteristics of thin-film circuit elements.

#### Statement of the Problem

Design techniques for the frequency response in distributed-parameter networks are developed by approximating the response of the system by a set of asymptotes. This approximation technique permits rapid calculation of the response with an error not exceeding 20 percent. In general, the error introduced by the approximation is less than 1 percent. Successive tasks in developing an asymptotic description of distributed-parameter systems will be as follows:

- (1) Obtain high and low frequency asymptotes for phase and amplitude by judicious approximations of transcendental functions at the extremes of their range.
- (2) Obtain break-points for asymptotes, i.e., critical frequencies separating the regions of high and low frequency response. High-frequency asymptotes are valid at a frequency exceeding the breakpoint and the low-frequency asymptotes are valid at frequencies below the breakpoint.
- (3) Compute order-magnitude estimates of the error introduced by employing asymptotes for both phase and magnitude.
- (4) Assess the validity of asymptotic approximations by a comparison of estimated and computed error.
- (5) Devise a scheme for evaluating the least number of functions to describe a three-terminal and eventually four-terminal networks.
- (6) Present representative examples illustrating the range and effectiveness of method.

Of the above tasks, (1) and (2) have been successfully completed; tasks (4) and (5) have yielded some interesting results; while tasks (3) and (6) will be the topics of a future investigation.

#### Critical Review of Related Investigation

Asymptotic techniques are extensively used in lumped parameter networks (refs. 1, 2, 3, 4). To adapt these techniques to distributed-parameter networks, it is instructive to review the pertinent properties of lumped-parameter asymptotes. The network response is described in terms of two singularly inter-related components, the amplitude ratio (gain) and the phase-difference, with respect to the network response at certain

reference frequencies. Approximations of these two functions facilitate rapid construction of a graphical presentation sufficiently accurate for engineering calculations.

Lumped-parameter network functions can generally be expressed as a ratio of polynomials  $M(s)/N(s)$  in non-negative powers of the frequency  $s = j\omega$ , where  $M(s) = \sum_i M_i s^i = M_0 \prod_i (s - s_i) = M_0 \prod_i (1 - \tau_i s)$  and similarly for  $N(s)$ . Consider now three distinct cases:

(1) If the network has only a finite number  $R$  and  $C$  (or  $R$  and  $L$ ) elements there are, in general, a number of real finite roots  $s_i = 1/\tau_i = \omega_i$  and they yield a set of break frequencies. The intersections of frequency asymptotes as obtained on a log-log plot are given in Table I. The asymptotes consist of straight lines with a discontinuity in the derivative of the magnitude and in the phase at the breakpoint. Correction factors between actual and asymptotic values for frequencies near the break-frequency are given in Table I where the  $\pm$  depend on  $\omega_i$ . On a log-plot both the product of factors  $(1 - \tau_i s)$  as well as the errors add linearly, permitting rapid compensation of magnitude and phase of  $M(s)$  as well as an estimate of the corresponding error.

(2) If the network contains  $R$ ,  $C$ , and  $L$  elements, complex roots  $s_i$  result in conjugate pairs. Asymptotes and error correction curves can be constructed using essentially the same technique for expression of the type  $1 + as + bs^2$ .

Table I.- Actual Values and Errors for RC Network

	Range of Validity		Log Magnitude	Phase
ACTUAL	$\omega \ll \omega_i$		0	0
	$\omega = \omega_i$		$\ln \sqrt{2}$	$\pi/4$
	$\omega \gg \omega_i$		$\ln (\omega/\omega_i)$	$\pi/2$
ERROR	$\epsilon_1 (\omega)$	$(\omega \leq \omega_1)$	$+1/2 \ln (1 + \tau_1^2 \omega^2)$	$\pm \tan^{-1} (\tau_1 \omega)$
	$\epsilon_2 (\omega)$	$(\omega \geq \omega_1)$	$1/2 \ln (1 + \frac{1}{\tau_1^2 \omega^2})$	$-\tan^{-1} (\frac{1}{\tau_1 \omega})$

(3) If the network contains only energy storing element, either L or C or both, roots occur generally at zero of infinity and judicious extension of the above techniques are required. For film-type circuits only R and C elements are employed, therefore reference is made in Table I to a circuit with an R and a C element and unit response at  $\omega = 0$ .

## REVIEW OF APPLICABLE TWO-PORT THEORY

### Characterization of Frequency Dependent Two-Ports

Two-ports are characterized by six conventional sets of circuit parameters, known as a, b, g, h, y, or z-parameters (Table II). Each of these sets contain four parameters; hence, twenty-four parameters need to be considered for a complete investigation of a given two-port. Introduction of concept of deviations (refs. 5, 6) furnishes a network analyst with the following advantages:

(1) Basic functions characterizing a two-port reduce in number from twenty-four to six. For a passive and symmetrical network this number reduces further to four. Hence the concept of deviation yields a systematic saving of effort in network analysis.

(2) A circuit designer desiring sufficiently accurate design information from asymptotic approximations can easily get that by drawing only six asymptotes, which reduce to four asymptotes for passive and symmetrical networks.

(3) Error analysis in design techniques is similarly reduced to a minimum. Because of these advantages associated with deviations in network analysis, frequency-dependent two-ports will be characterized by deviations. Table II shows conventional network functions in terms of deviations. To make this presentation self-contained and useful to engineers without background in this relatively new tool, the rules governing deviations will be briefly reviewed. Deviations describe multiport networks and are defined in several alternative ways:

- (a) In terms of cut-sets and tie-sets based on topological properties of networks.
- (b) In terms of partial differentials. This approach provides a useful definition, but has the advantage that only ratios of deviations, not deviations themselves, are defined. In network applications only ratios of deviations are needed, since a partial derivative can always be expressed as the ratio of two deviations.
- (c) In terms of derivatives with respect to a new parameter  $t$ , which need never be specified. If

all the partial derivatives of a system are referred to some variable  $t$ ,

$$\frac{\partial x}{\partial u} \Big|_y = \frac{\partial x}{\partial t} \Big|_y \Big/ \frac{\partial u}{\partial t} \Big|_y \triangleq \frac{(x,y)}{(u,y)}$$

If only ratios of deviations are considered, then  $t$  need not be stated explicitly. This permits the definition of the deviation  $(x,y)$  as a partial derivative with respect to  $t$ .

TABLE II.- NETWORK PARAMETERS IN DEVIATION NOTATION

Deviation	Symbol	Subscripts				Determinant
		11 or i	12 or r	21 or f	22 or 0	
$(v_o, i_o)$	(a)	$\frac{(g)}{(a)}$	$-\frac{(y)}{(a)}$	$\frac{(z)}{(a)}$	$-\frac{(h)}{(a)}$	$-\frac{(b)}{(a)}$
$(i_i, v_i)$	(b)	$\frac{(h)}{(b)}$	$-\frac{(y)}{(b)}$	$\frac{(z)}{(b)}$	$-\frac{(g)}{(b)}$	$-\frac{(a)}{(b)}$
$(v_i, i_o)$	(g)	$\frac{(z)}{(g)}$	$-\frac{(b)}{(g)}$	$\frac{(a)}{(g)}$	$\frac{(y)}{(g)}$	$\frac{(h)}{(g)}$
$(i_i, v_o)$	(h)	$\frac{(y)}{(h)}$	$\frac{(b)}{(h)}$	$-\frac{(a)}{(h)}$	$\frac{(z)}{(h)}$	$\frac{(g)}{(h)}$
$(v_i, v_o)$	(y)	$\frac{(h)}{(y)}$	$-\frac{(b)}{(y)}$	$-\frac{(a)}{(y)}$	$\frac{(g)}{(y)}$	$\frac{(z)}{(y)}$
$(i_i, i_o)$	(z)	$\frac{(g)}{(z)}$	$\frac{(b)}{(z)}$	$\frac{(a)}{(z)}$	$\frac{(h)}{(z)}$	$\frac{(y)}{(z)}$

The symbol  $(x,y)$  is called the deviation of  $x$  with respect to  $y$ .

To illustrate, conventional  $h_i$  in Figure 1

$$h_i = \frac{\partial v_1}{\partial I_1} \Big|_{v_0}$$

becomes in terms of deviation notation

$$h_i = \frac{(V_1, V_0)}{(I_1, V_0)}$$

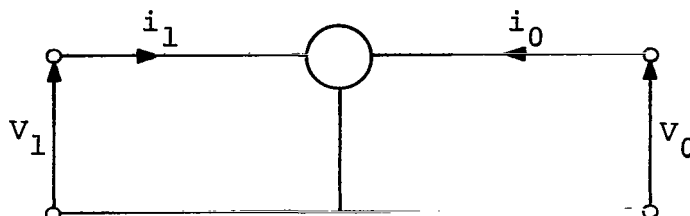


Figure 1.- Sign conventions on currents and voltages in a two-port

Similarly, other basic rules (ref. 5) have been derived:

(1) A three-terminal two port is characterized by six deviations (a), (b), (g), (h), and (z) defined in Table II.

(2) Table II shows network parameters expressed as ratios of deviations.

(3) Anti-commutation, multiplicative, associative laws hold for deviations:

$$(x, y) = (-y, x)$$

$$(Kx_1, x_2) = K (x_1, x_2)$$

$$(x_1 + x_2, x_3) = (x_1, x_3) + (x_2, x_3)$$

$$(4) (x_1 x_2, x_3) = x_1 (x_2, x_3) + x_2 (x_1, x_3)$$

(5) Any four variables  $x_1, x_2, x_3, x_4$  and constrained by  $x_1 = f(x_3, x_4)$  and  $x_2 = f(x_3, x_4)$  are related by

$$(x_1, x_2) (x_3, x_4) + (x_2, x_3) (x_1, x_4) + (x_3, x_1) (x_2, x_4) = 0$$

This relationship reduces, for a two-port, to a constraint (a) (b) + (y) - (g) (h) = 0

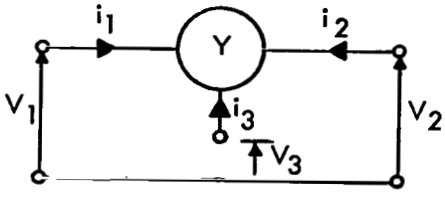
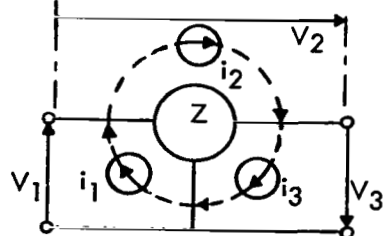
(6) An additional deviation (f) is often defined

$$(v_o - v_1, i_o + i_1) = (f) = (g) + (h) - (a) - (b)$$

The significance of (f) is seen from the indefinite matrix (ref. 7), either [Y] or [Z], compared in Table III.



TABLE III.- TERMINAL AND PORT FORMULATION OF NETWORKS

	
$[I] = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix}, \quad [V] = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$ $[I] = [Y][V]$	$[I] = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix}, \quad [V] = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$ $[V] = [Z][I]$
$[Y] = \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{bmatrix}$	$[Z] = \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{bmatrix}$
$[Y] = \frac{1}{y} \begin{bmatrix} h & -b & b-h \\ -a & g & a-g \\ a-h & b-g & f \end{bmatrix}$	$[Z] = \frac{1}{z} \begin{bmatrix} g & b-g & -b \\ a-g & f & b-h \\ -a & a-h & h \end{bmatrix}$
<p> <math>(v_2, i_2)</math>  <math>(i_1, v_1)</math>  <math>(v_1, i_2)</math>  <math>(i_1, v_2)</math>  <math>(i_3, v_2 - v_1)</math>  <math>(v_1, v_2)</math>  <math>(i_1, i_2)</math> </p>	<p> <math>a</math>  <math>b</math>  <math>g</math>  <math>h</math>  <math>f</math>  <math>y</math>  <math>z</math> </p> <p> <math>(v_3, i_3)</math>  <math>(i_1, v_1)</math>  <math>(i_1, v_3)</math>  <math>(v_1, i_3)</math>  <math>(v_2, i_1 - i_3)</math>  <math>(v_3, i_1)</math>  <math>(i_3, i_1)</math> </p>

Evaluation of the elements of the matrices is accomplished by grounding a terminal for the [Y] matrix or opening a port for the [Z] matrix. For example,  $y_{11}$ ,  $y_{12}$ ,  $y_{21}$ , and  $y_{22}$  can be evaluated by grounding the terminal 3.

Referring to Table II

$$y_{11} = h/y \quad y_{12} = -b/y$$

$$y_{21} = -a/y \quad y_{22} = g/y$$

For convenience of writing deviations, the notation  $h/y$  instead of  $(h)/(y)$  is employed when no ambiguity exists.

Since the sum of the elements in any row or column of [Z] or [Y] is zero, the rest of the elements are found easily. Matrix relationship can now be written, in terms of deviations as given in Table III.

#### Distributed Parameter Networks

The deviations of a simple distributed parameter network shown in Table IV are to be derived by proceedings as follows:

(1) Only deviations (a), (g), and (f) for the common capacitor (cc) have to be computed, since for a passive network  $a = b$  and for a symmetrical network  $g = h$ .

(2) The deviations for the capacitor input (ci) and capacitor output (co) configurations are obtainable from Table IV using the auxiliary variable

$$(f) = (g) + (h) - (a) - (b)$$

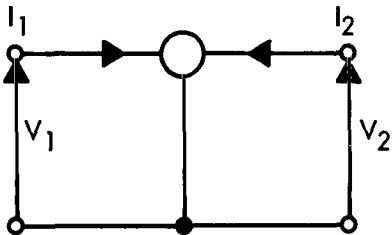
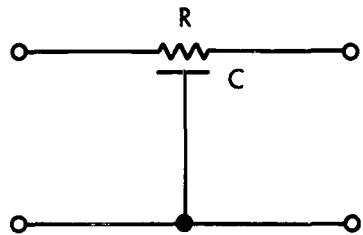
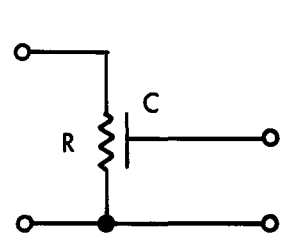
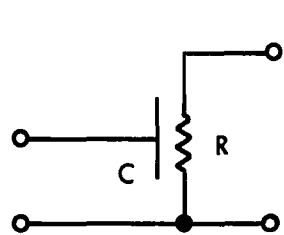
Thus  $(a) = (b)$  and  $(g) = (h)$  yields

$$(g) - (a) = (h) - (b) = (f)/2$$

(3) While it is possible to compute deviations directly from cut-sets and tie-sets, the deviations (a), (g) and (f) listed in Table III can also be obtained from the matrix development given below.

(4) For the selection of deviations, an arbitrary constant may multiply any deviation. This constant is selected such that  $(a) = (b) = (g) = (h) = 1$  at low frequencies for the (cc) network in Table III.

TABLE IV.- DEVIATIONS FOR DISTRIBUTED - PARAMETER  $\overline{RC}$  NETWORKS

TWO-PORT	COMMON CAPACITANCE (CC)	CAPACITANCE OUTPUT (CO)	CAPACITANCE INPUT (CI)
			
(a) = (b)	$\theta / \sinh \theta$	$\theta \tanh \theta / 2$	$\theta \tanh \theta / 2$
(g)	$\theta \coth \theta$	$2 \theta \tanh \theta / 2$	$\theta \coth \theta$
(h)	$\theta \coth \theta$	$\theta \coth \theta$	$2 \theta \tanh (\theta / 2)$
(y)	$R$	$R$	$R$
(z)	$\theta^2 / R$	$\theta^2 / R$	$\theta^2 / R$
(f)	$2 \theta \tanh \theta / 2$	$\theta \coth \theta$	$\theta \coth \theta$

Matrix Development:

$$\begin{bmatrix} V \\ I \end{bmatrix} = \frac{\theta}{R} \begin{bmatrix} \coth\theta & -\operatorname{csch}\theta \\ -\operatorname{csch}\theta & \coth\theta \end{bmatrix} = \frac{1}{(y)} \begin{bmatrix} (h) & -(b) \\ -(a) & (g) \end{bmatrix}$$

where  $\theta = \sqrt{i\omega CR}$   
 $R = \text{total resistance of the film} = r_o \ell$   
 $C = \text{total capacitance of film} = C_o \ell$   
 $\ell = \text{length of the film}$

This matrix describes the cc configuration and is divided from incremental circuit (Figures 2 and 3) such that for  $0 < x < \ell$

$$\frac{dV}{dx} = -Ir_o, \quad \frac{dI}{dx} = -Vi\omega C_o$$

$$\text{or } \frac{d^2V}{dx^2} = V\alpha, \quad \frac{d^2I}{dx^2} = I \quad \text{where } \alpha = \theta/\ell$$

Yielding

$$V = A e^{\alpha x} + B e^{-\alpha x}$$

$$I = C e^{\alpha x} + D e^{-\alpha x}$$

where A, B, C and D are constants to be evaluated from the boundary conditions at  $x = 0$   $V = V_1$ ,  $I = I_1$  and at  $x = \ell$ ,  $V = V_2$ ,  $I = I_2$

Evaluating C and D,

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{1}{R} \begin{bmatrix} \theta \coth \theta & -\theta/\sinh \theta \\ -\theta/\sinh \theta & \theta \coth \theta \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

Hence  $(y) = R$ ;  $(h) = (g) = \theta \coth \theta$ ;  $(a) = (b) = \theta/\sinh \theta$  which yields  $(z) = \theta^2/R$ .

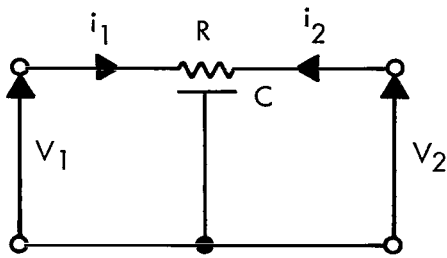


Figure 2.- Distributed  
Parameter RC  
Network

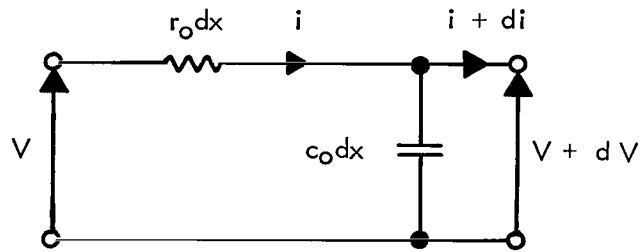


Figure 3.- Incremental Model of  
RC Network

### Asymptotes for Lumped Parameter Network and Deviations

Frequency domain analysis is a powerful tool in system design and is used frequently in association with the asymptotic techniques (refs. 1, 2, 3, 4).

- (1) To determine the stability of a system (Nyquist-Criteria)
- (2) To determine gain - and phase-margin of a given system. Alternatively, to design a system with given gain and phase-margin.
- (3) To determine the frequency response characteristic of an equalizer often used to improve on the response characteristics of a given system.
- (4) To determine the close-loop response from a given open-loop response with the aid of M and N circles and Nichols charts (ref. 2).

Frequency response of lumped parameter systems can be described in several alternative ways:

- (1) A polar plot of the frequency response function
- (2) Two complementary curves: one for amplitude of the function vs frequency, another for the phase of the function vs frequency
- (3) Two complementary curves: one for logarithm of the magnitude of the frequency response function vs logarithmic frequency scale and another for phase of function vs logarithm of the frequency.

In the discussion to follow, attention is focused on frequency response. Specifically, properties which can be conveniently expressed by deviations are:

- (1) The asymptotes may be drawn with a minimum of computation.

(2) If the true curve is needed, it may be formed by correcting the asymptotic curve.

(3) Transfer and driving point functions of lumped parameter networks are rational functions in frequency which can be factored.

Four basic types of frequency functions are used:

- (1)  $K$  (a constant)
- (2)  $(i\omega) \pm n$
- (3)  $(i\omega T + 1) \pm n$
- (4)  $[(i\omega/\omega_n)^2 + 2(i\omega/\omega_n) + 1] \pm n$

Because the logarithm of a function in factored form is the sum or logarithms of factors, a composite curve for a frequency response function with many factors can be readily obtained from curves corresponding to four basic types mentioned.

The logarithm of a constant is just another constant and is invariant with  $\omega$ . Thus, a plot of  $20 \log K$  vs  $\log \omega$  is a horizontal line. The phase angle is 0 degrees for all frequencies.

The magnitude curve of the factor  $(i\omega)^{\pm n}$  is a straight line having a slope of  $\pm 20 n$  db/decade passing through 0 - db at one radian per second. The phase angle is  $\pm n\pi/2$  at all frequencies.

Magnitude asymptote for the third type of factor consists of two parts - a straight line segment coinciding with the 0 db line and another line segment with a slope of  $\pm n$  db/decade. The point of intersection of these  $\omega_0$  asymptotes, known as the break-point, is given by  $\omega T = 1$ . Phase asymptotes for this type of function are given by two straight lines: one coincides with the 0 radian line and another coincides with  $\pm n\pi/2$  radians line. Breakpoints for the phase-asymptotes are made to coincide with those of the amplitude asymptotes.

The fourth type of factor can also be described by asymptotes. At low frequencies, the magnitude is unity and the asymptote is a horizontal line at 0 db. At high frequencies, the magnitude is  $(\omega/\omega_n)^{2n}$  and the asymptote is a straight line with slope of  $\pm 20n$  db/decade. The asymptotes intersect on the 0 db axis at  $\omega = \omega_n$ , which is the break frequency. Phase asymptotes of this type of factor is again represented by two straight lines: one coincides with 0-radian axis (for  $\omega \leq \omega_n$ ) and the other coincides with the line passing through  $\pm n$  (for  $\omega \geq \omega_n$ ). Magnitude and phase asymptotes for different types of factors are given in most textbooks (refs. 3,4).

It should be noted that factors of the first and second types are accurately represented by asymptotes while the third and fourth types of factors accrue errors from their asymptotic representations. Asymptotic representation of the magnitude of third type of function always leads to positive errors (asymptotic values greater than the actual values) while errors in the fourth type of function depend on the damping ratio  $\xi$ . The true magnitude curve of this type of factor depends on the damping ratio which is denoted by  $\xi$ . For  $\xi < .707$ , error is always negative while for  $\xi > .707$ , error is positive. Phase asymptotes for the third and fourth type of functions always lead to positive errors, in the latter case error being a function of  $\xi$ .

### Non-Redundant Network Functions

The primary motivation for introducing the concept of deviations is to minimize the number of network functions to completely describe a network. For effective presentation of network driving-point and transfer functions, the next important step is to enumerate non-redundant network functions. In Tables V and VI, a three-terminal network is considered and unique driving-point and transfer functions, resulting from various configurations of the network are listed.

Driving-point Functions.- To evaluate driving-point functions consider the following:

(1) Exchange of sources (voltage and current) leads to reciprocal driving-point functions, impedance and admittance, which are not listed separately

(2) If the network is passive  $(a) = (b)$  since the appropriate cut-sets and tie-sets defining deviations are equal for passive networks. The reciprocity law in network theory also indicates  $(a) = (b)$

(3) If the network is symmetrical  $(g) = (h)$ , since an interchange of terminals, say 1 and 2, transforms  $(g)$  into  $(h)$  and vice versa. This interchange leaves the external characteristics invariant and generates no novel network functions. From Table V unique driving point functions are:

$(g)/(z)$   $(y)/(h)$ ,  $(f)/(z)$  and  $(y)/(f)$  for the symmetrical network,

and

$(g)/(z)$ ,  $(y)/(h)$ ,  $(h)/(z)$ ,  $(y)/(g)$ ,  $(f)/(z)$ ,  $(y)/(f)$ , for the asymmetrical network.

Transfer Functions.- For the evaluation of non-redundant transfer functions considerations (2) and (3) in Driving-Point functions above hold but exchange of sources leads to new transfer functions. Additional considerations to be made are:

TABLE V.- UNIQUE DRIVING-POINT FUNCTIONS

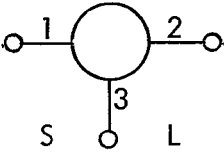
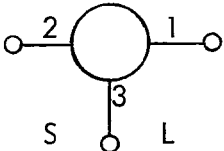
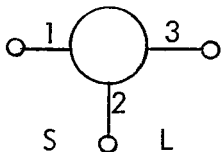
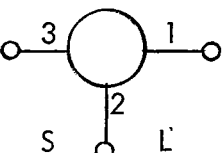
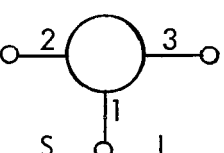
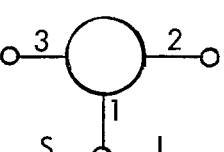
S Generator L-Output	Impedance $v_{in}/i_{in}$		Number of unique driving-point functions		
	L open	L short	Active	Passive	Symmetrical
	$(g)/(z)$	$(y)/(h)$	2	2	2
	$(h)/(z)$	$(y)/(g)$	2	2	1
	$(f)/(z)$	$(y)/(h)$	1	1	1
	$(h)/(z)$	$(y)/(f)$	1	1	0
	$(f)/(z)$	$(y)/(g)$	0	0	0
	$(g)/(z)$	$(y)/(f)$	0	0	0



TABLE VI.- UNIQUE TRANSFER FUNCTIONS

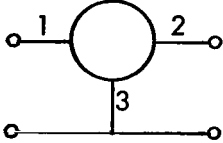
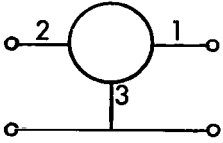
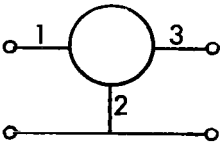
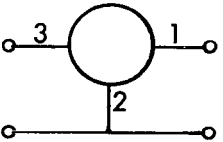
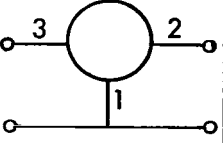
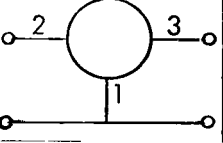
Drive-input	$i_{in}$		$e_{in}$		Number of unique transfer functions		
Output	$e_o/i_i _{i_o}$	$i_o/i_i _{e_o}$	$e_o/e_i _{i_o}$	$i_o/e_i _{e_o}$	Active	Passive (a)=(b)	Symmetrical (1) $\longleftrightarrow$ (2)
	$(a)/(z)$	$-(a)/(h)$	$(a)/(g)$	$-(a)/(y)$	4	4	3
	$(b)/(z)$	$-(b)/(g)$	$(b)/(h)$	$-(b)/(y)$	4	0	0
	$\frac{(h)-(a)}{(z)}$	$\frac{(h)-(a)}{(h)}$	$-\frac{(h)-(a)}{(f)}$	$-\frac{(h)-(a)}{(y)}$	4	4	4
	$\frac{(h)-(b)}{(z)}$	$\frac{(h)-(b)}{(f)}$	$\frac{(h)-(b)}{(h)}$	$-\frac{(h)-(b)}{(y)}$	4	0	0
	$-\frac{(a)-(g)}{(z)}$	$\frac{(g)-(a)}{(f)}$	$-\frac{(a)-(g)}{(g)}$	$\frac{(a)-(g)}{(y)}$	4	4	0
	$\frac{(g)-(b)}{(z)}$	$\frac{(g)-(b)}{(g)}$	$\frac{(g)-(b)}{(f)}$	$-\frac{(g)-(b)}{(y)}$	4	0	0

TABLE VII.- DEVIATIONS OF TWO-PORT CONFIGURATIONS RESULTING  
FROM INTERCHANGE OF TERMINALS OR PORTS

Terminal	123	312	231	213	132	321
Port	123	312	231	321	213	132
z	z	z	z	-z	-z	-z
y	y	y	y	-y	-y	-y
h	h	f	g	-g	-h	-f
g	g	h	f	-h	-f	-g
f	f	g	h	-f	-g	-h
b	b	h-a	g-a	-a	b-h	b-g
a	a	h-b	g-b	-b	a-h	a-g

- (1) For a symmetrical network

$$v_{out}/v_{in} \Big|_{i_{out}} = - i_{out}/i_{in} \Big|_{v_{out}}$$

- (2) For a passive network reciprocity holds. From Table VI unique transfer functions are:

$$(a)/(g), (a)/(y), (a)/(z), \frac{(h) - (a)}{(f)}, \frac{(h) - (a)}{(z)}, \frac{(h) - (a)}{(h)} \text{ for passive, symmetrical networks.}$$

- (3) Transfer functions for passive, unsymmetrical network are:

$$(a)/(g), (a)/(y), (b)/(h), (a)/(z), \frac{(h) - (a)}{(f)}, \frac{(h) - (a)}{(y)}, \frac{(h) - (b)}{(h)}, \frac{(h) - (a)}{(z)}, \frac{(a) - (g)}{(g)}, \frac{(a) - (g)}{(g)}, \frac{(g) - (b)}{(f)}, \frac{(a) - (g)}{(z)}$$

Unique driving-point and transfer functions for an active network are also obtainable from Tables V and VI assuming  $(a) \neq (b)$ .

Three-terminal Network.- Transformation of deviations (ref. 8) resulting from rotation or reflection of a three-terminal network are summarized in Table VII using the notation of Table III. Note that Table III defines deviations in terms of both, terminal and port variables. Terminals and ports transform differently under reflection, but the same physical transformation yields the same deviations, whether expressed in terminal or port notation. Table III in conjunction with Table VII consolidates and restates from a somewhat different point-of-view much of the information presented in this chapter.

#### Representative Distributed Parameter Two-Ports

Typical values for  $a, b, g, h, f, y, z$  for distributed-parameter networks are listed in Table VIII for the symmetric two-ports and in Table IX for asymmetric two-ports (ref. 9). The distributed-parameter RC network is given for comparison with Table IV.

For the transistor (ref. 10) the variables of the two-port are taken as currents and minority-carrier concentration densities. The resultant parameters are  $\tau_p$ , the minority-carrier lifetime, the diffusance

$$H = eAD_p/W, \text{ and } \tau_w = w^2/D_p$$

is a time constant defined by  $w$ , the width of the base, and  $D_p$ , the diffusion constant of minority carriers. The effective area is  $A$  and the carrier charge is  $e$ .

TABLE VIII.- SYMMETRIC DISTRIBUTED PARAMETER TWO-PORTS

Deviation	Thin-film RC Network	Minority-Carrier Diffusion Transistor
$a = b$	$\text{csch } (sRC)^{1/2}$	$\text{csch } (s\tau_w + \tau_w/\tau_p)^{1/2}$
$g = h$	$\text{coth } (sRC)^{1/2}$	$\text{coth } (s\tau_w + \tau_w/\tau_p)^{1/2}$
$f$	$\tanh \frac{1}{2}(sRC)^{1/2}$	$\tanh \frac{1}{2}(s\tau_w + \tau_w/\tau_p)^{1/2}$
$y$	$(sC/R)^{-\frac{1}{2}}$	$H(1 + s\tau_p)^{-\frac{1}{2}}$
$z$	$(sC/R)^{1/2}$	$(1 + s\tau_p)^{1/2}/H$

In Table IX the parameter  $c$  is a measure of the taper, for  $c = 0$  the uniform network results.

TABLE IX.- ASSYMETRIC DISTRIBUTED PARAMETER TWO-PORT

Deviation	Tapered RC thin-film network
$a = b$	$\text{csch}(c^2 + sCR)^{1/2}$
$g$	$e^c \text{coth}(c^2 + sCR)^{1/2} - c/(c^2 + sCR)^{1/2}$
$h$	$e^{-c} \text{coth}(c^2 + sCR)^{1/2} + c/(c^2 + sCR)^{1/2}$
$y$	$R(c^2 + sCR)^{-1/2} e^c$
$z$	$R^{-1}(c^2 + sCR)^{1/2} e^{-c}$

#### FREQUENCY ASYMPTOTES IN DISTRIBUTED-PARAMETER STRUCTURES

##### Low and High Frequency Asymptotes for Deviations

Magnitude and phase asymptotes for the deviations are obtained as the limiting values of the function as  $\theta \rightarrow 0$  and  $\theta \rightarrow \infty$ . As an illustration, consider

$$g(\theta) = \theta \coth \theta$$

where  $g(\theta)$  is used to denote  $(g) = (g(\theta))$  consider  $g(\theta)$  as  $\theta \rightarrow 0$ ,  $g(0) = 1 + \theta^2/3 - \theta^4/45 = \dots \rightarrow 1$ . Similarly, at  $\theta \rightarrow \infty$

$$g(\infty) \rightarrow \theta$$

High and low-frequency magnitude asymptotes for deviations  $a$  and  $f$  are derived similarly. Table X gives the asymptotes for the three deviations.

Corresponding phase asymptotes are obtained by evaluating phases of the functions at  $\theta \rightarrow 0$  and at  $\theta \rightarrow \infty$ . For example, for  $g = \theta \coth \theta$ , as  $\theta \rightarrow 0$  then

$$\coth \theta = 1$$

$$\text{and } g = 0$$

Similarly for  $g(\theta)$  at  $\theta \rightarrow 0$ , the angle  $\angle g(0) = 0$ .

Again, since  $g(\infty) = \theta = j\omega CR$ , hence

$$\angle g = \pi/4$$

Phase asymptotes for the deviations are shown in Table X with  $(y) = R$  and  $(z) = \theta/R$ .

TABLE X.- ASYMPTOTES FOR DEVIATIONS OF  $\overline{RC}$  NETWORK

$\theta = \sqrt{j\omega CR} = \sqrt{jx}$		Magnitude $ f(x) $		Phase $\angle f(x)$	
Deviations	$f(x)$	$x \rightarrow 0$	$x \rightarrow \infty$	$x \rightarrow 0$	$x \rightarrow \infty$
$a = b$	$(jx)^{1/2} \text{csch}(jx)^{1/2}$	1	$(4x)^{1/2} \exp \left[ -(x/2)^{1/2} \right]$	0	$\pi/4 - (x/2)^{1/2}$
$g = h$	$(jx)^{1/2} \coth(jx)^{1/2}$	1	$x^{1/2}$	0	$\pi/4$
$f$	$2(jx)^{1/2} \tanh(jx/4)^{1/2}$	$x$	$2x^{1/2}$	$\pi/2$	$\pi/4$

Note that the deviations listed in Tables IV, VIII, and X are only consistent within a single table, but otherwise are not unique.

### Break-Points of Asymptotes

The intercept of the amplitude asymptotes is termed the break-point. The point of intersection is obtained by equating the two asymptotes. For example, the intercept  $x$  for the asymptotes of  $a$  is given by

$$\frac{\sqrt{4x}}{\sqrt{e^{x/2}}} = 1, \text{ or } x = \omega RC = 1.05$$

Break-points are shown in Table XI. Break-points for phase asymptotes are made to coincide with those of magnitude asymptotes.

TABLE XI.- BREAK-POINTS OF DEVIATIONS

Deviations	$f(0)$	$ f(0) $	$f(\infty)$	$ f(\infty) $	Breakpoint
$a = b$	1	1	$\frac{(2x)^{1/2}(1+j)}{\exp [x/2(1+j)]}$	$\frac{(4x)^{1/2}}{\exp (x/2)^{1/2}}$	$x = 1.05$
$g = h$	1	1	$(x/2)^{1/2}(1+j)$	$(x)^{1/2}$	$x = 1$
$f$	$jx$	$x$	$(2x)^{1/2} (1+j)$	$2(x)^{1/2}$	$x = 4$

Magnitude and phase asymptotes tabulated in Table X are plotted in Figures 4 and 5.

Asymptotes of the unique driving-point and transfer functions, Table VIII and IX, can be plotted from Figures 4 and 5 as shown in Figures 6 and 7.

### Discussion of Distributed-Parameter Network With Tapered Geometry

R-C distributed parameter networks with rectangular geometry and uniform resistive and dielectric films have been extensively investigated and their electrical characteristics are generally understood. Attention is now focused on tapered or non-uniform d-p networks because of certain desirable electrical characteristics not obtainable from rectangular, uniform d-p networks. Tapering can be made either geometrically or parametrically.

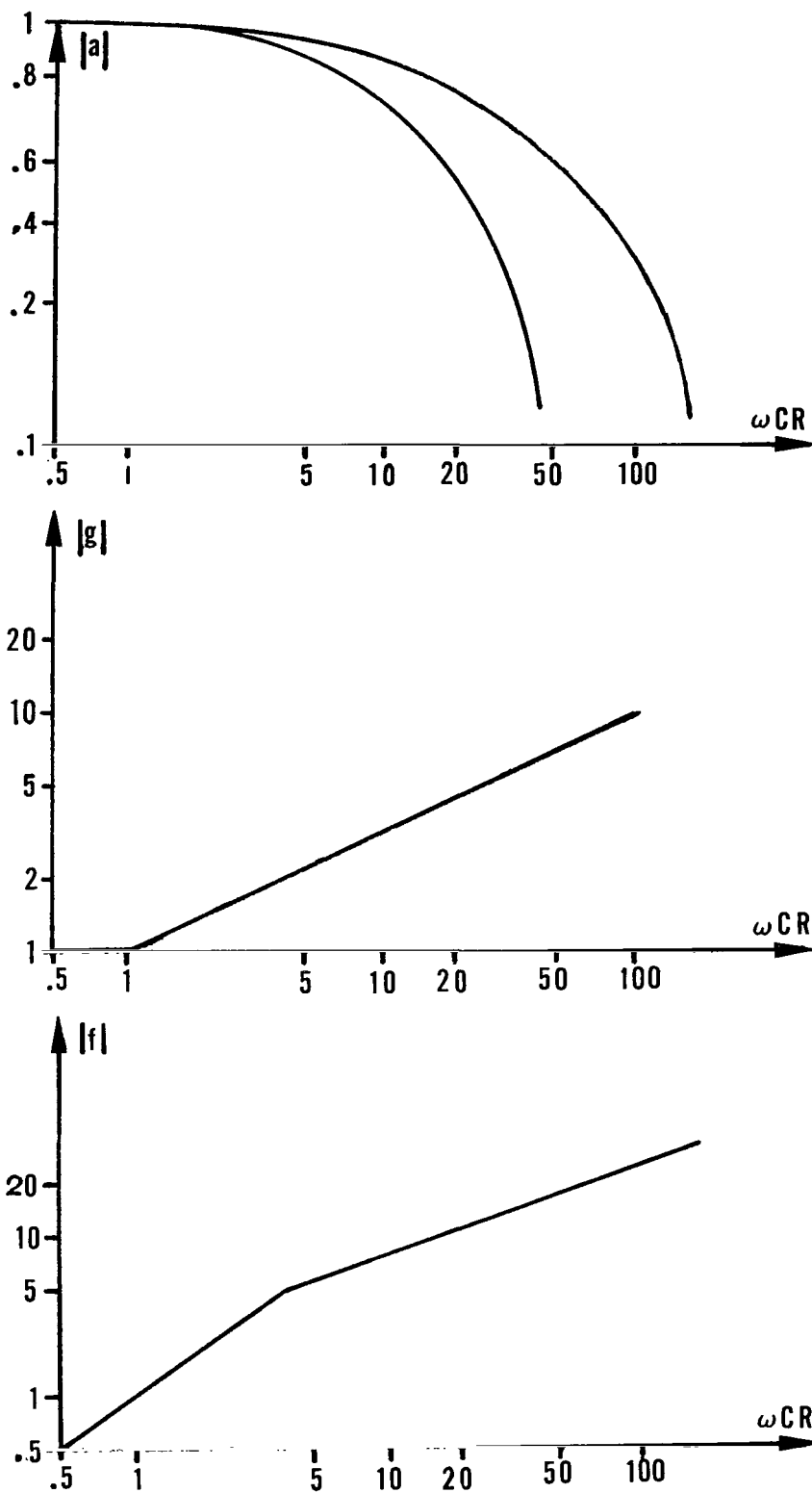


Figure 4.- Magnitude asymptotes for deviations

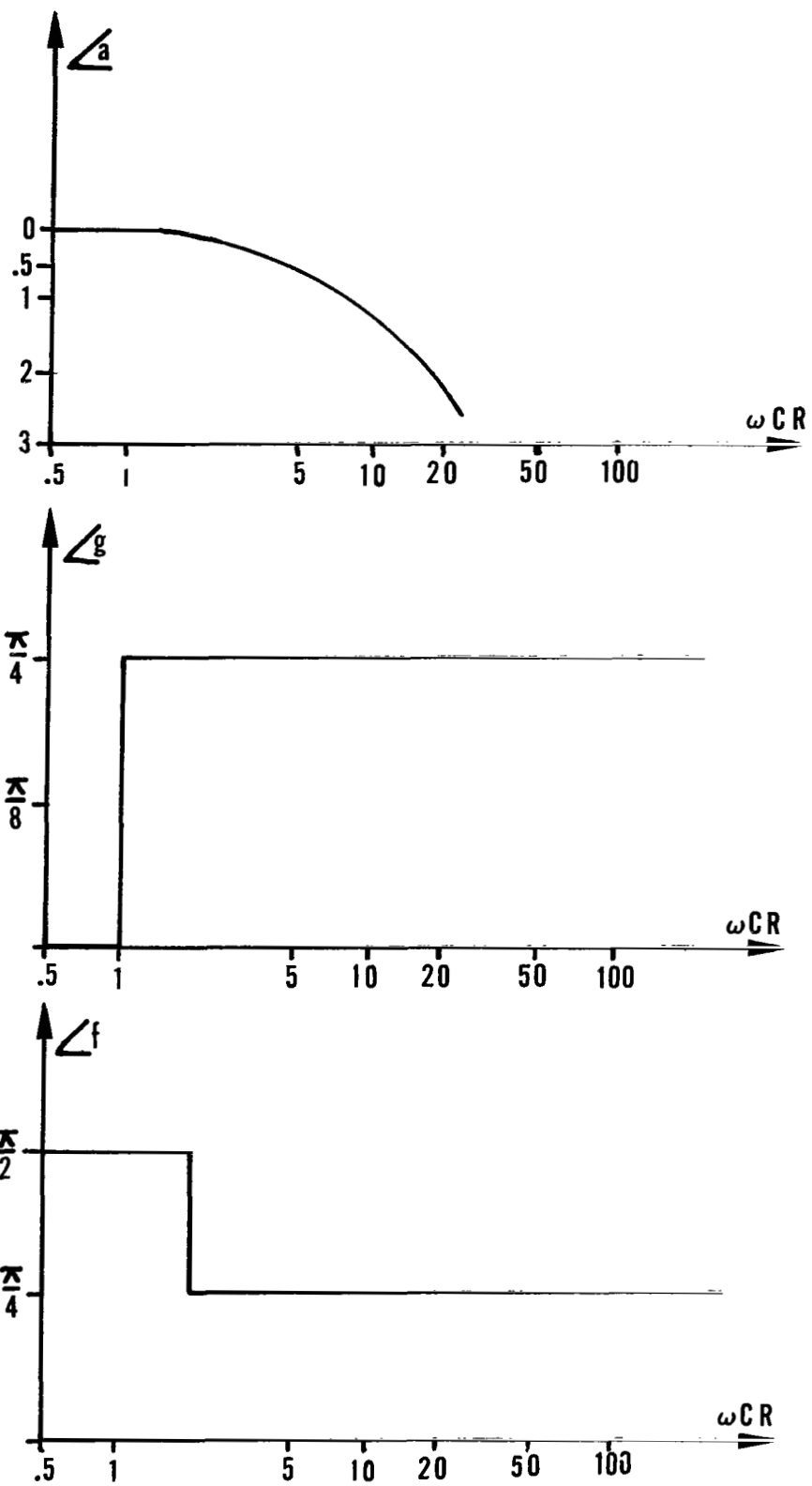


Figure 5.- Phase asymptotes for deviations



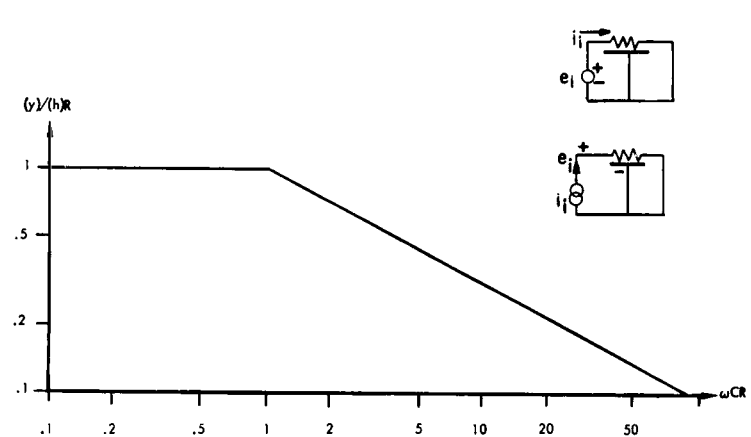
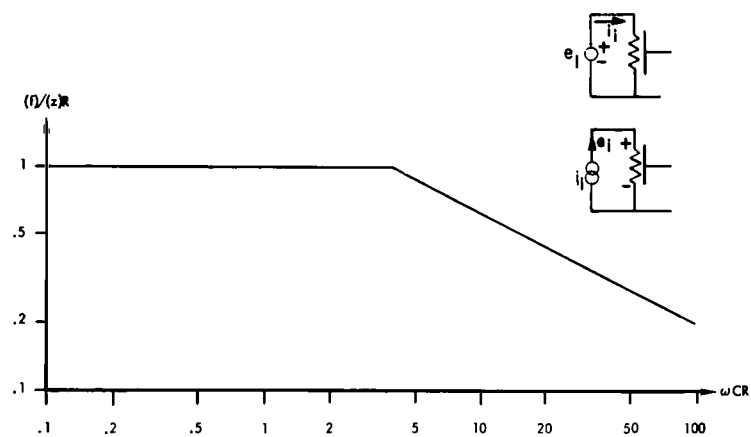
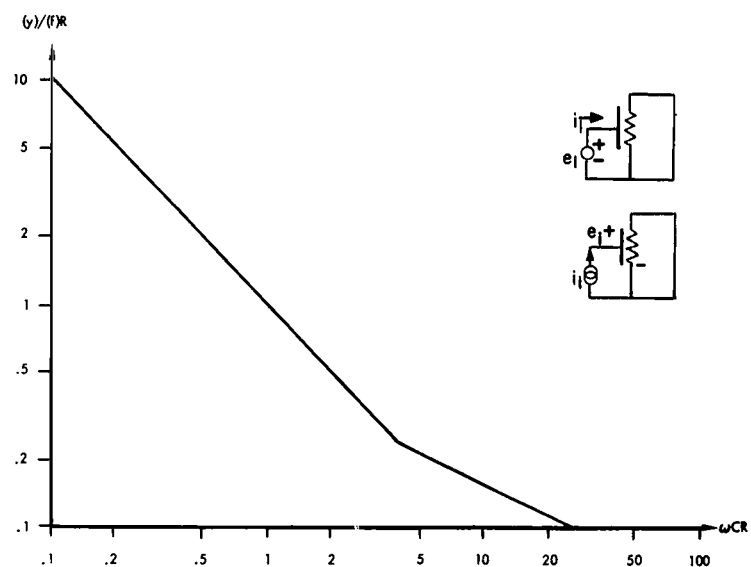
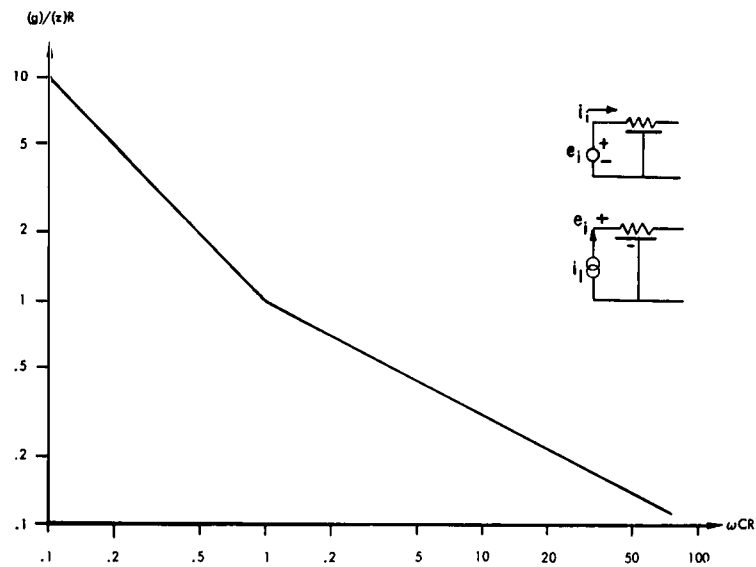


Figure 6.- Driving-point functions vs. normalized frequency

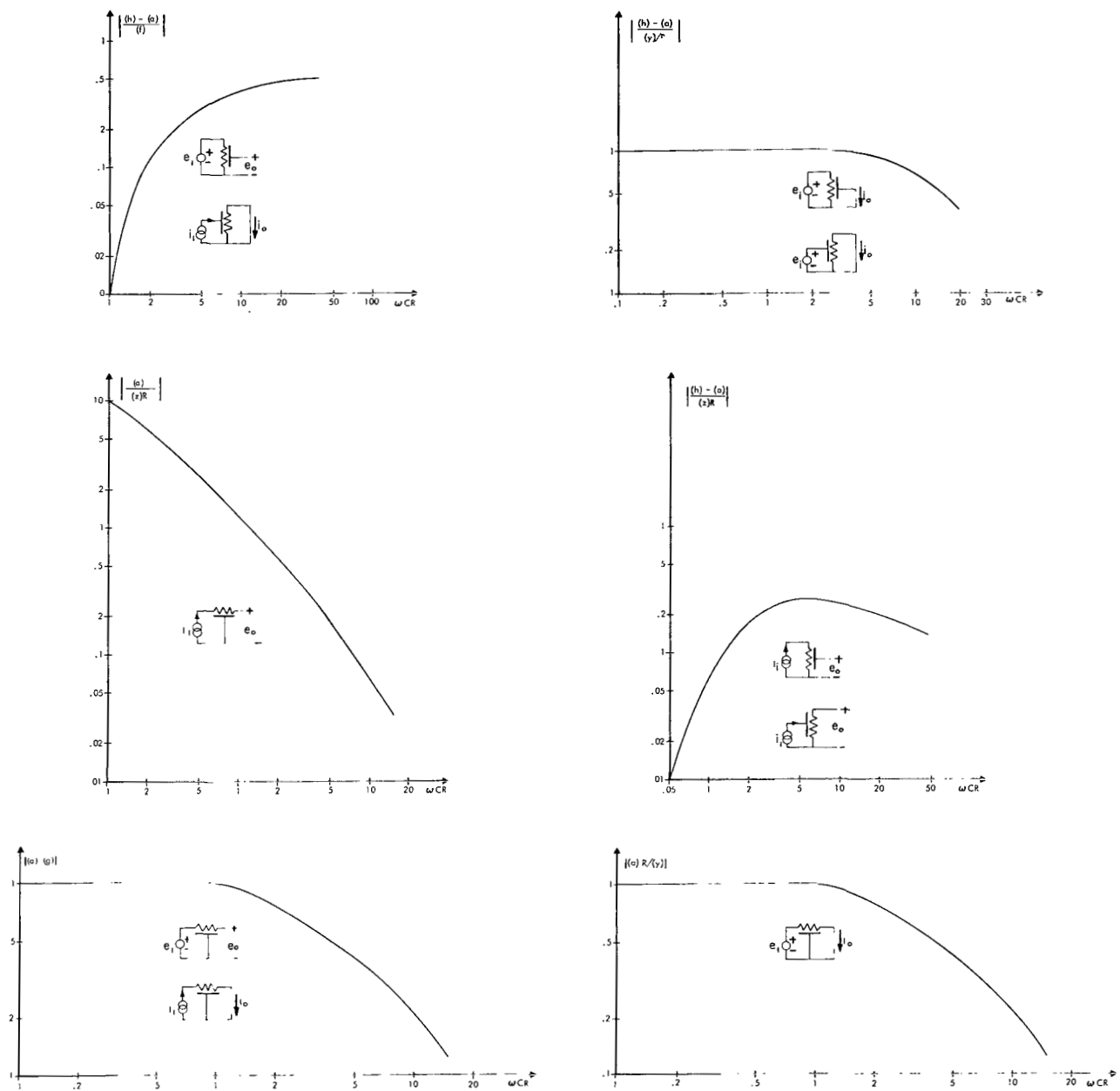


Figure 7.- Transfer functions vs. normalized frequency

Heizer (ref. 12) has shown that by proper geometric tapering distributed R-C networks with rational transfer functions can be realized, thus intensifying the use of these R-C networks (often characterized by irrational transfer function) in system design. Kaufman (refs. 13, 14) has shown that with exponentially tapered RC network null networks can be designed with roll-off characteristics faster than are obtainable with lumped RC and d-p RC circuits. Edson (ref. 15) has proposed tapered RC phase-shift oscillators which permit a reduction in the required levels of impedance and gain.

Analyses followed by Kaufman and Edson have been based upon parameter tapering only, using a rectangular geometry.

The case of general tapered RC network, either geometrically or parametrically, has not been solved. It has been shown by Castro and Happ (ref. 16) that distributed parameter networks with circular geometry, which possesses all the desirable characteristics (Edson, Kaufman) of tapered d-p networks, lend themselves to solution by a one-dimensional formulation. To aid in systems design with circular d-p network when applicable, asymptotic plots of short and open-circuit driving point and transfer functions and resulting errors are presented.

#### Derivation of Deviations For Tapered $\overline{RC}$ Networks

An R-C network with circular structure shown in Figure 8 is equivalent to a tapered network. Although solutions in terms of the indefinite admittance and as well as in terms of the indefinite impedance matrices, are known, a brief derivation in terms of deviations is given.

Edge effects are not considered; thus, the actual shape of the dielectric and conductive films are not considered as long as the films extend throughout the resistive film area. The outer and inner radii of the resistive film are  $W$  and  $W/n$  ( $n > 1$ ) and it is assumed to have a uniform  $r_o$  ohm per square. The dielectric film is assumed to have a uniform  $c_o$  farads per unit area. The matrices are obtained by lumped approximation of a distributed circuit as shown in Figure 9. In the Figure 9,  $dr$  is the elemental resistance of a ring of the film having inner and outer radii of  $l$  and  $l + dl$  is the elemental capacitance associated with it. The voltage and current down the film are given by

$$- \frac{dV}{dL} = r_o / 2\pi I \quad (1)$$

$$- \frac{dI}{dL} = i\omega c_o 2\pi V \quad (2)$$

noting that

$$\text{total resistance } R = \frac{r_o}{2\pi} \log n$$

$$\text{total capacitance } C = \frac{c_o \pi W^2}{n^2} (n^2 - 1)$$

and substituting

$$R/\log n = R', \quad C' = 2 n^2 C / (n^2 - 1), \quad \theta = (j\omega R' C')^{1/2}$$

Eqs. (1) and (2) can be combined to yield

$$d^2 V / dL^2 + (1/L) (dV/dL) - \theta^2 V / W^2 = 0 \quad (3)$$

with the solution

$$V = A I_0 (\theta / WL) + B K_0 (\theta / WL) \quad (4)$$

where  $I_0$  and  $K_0$  are modified Bessel functions of first and second kind respectively of zero order. Integrating Eq. (2) and applying boundary conditions (Figures 2 and 8)

$$I_1 = - \frac{\theta}{R'} \frac{1}{n} \left[ A I_1 (\theta / n) - B K_1 (\theta / n) \right] + D \quad (5)$$

$$I_2 = - (\theta / R') \left[ A I_1 (\theta) - B K_1 (\theta) \right] + D \quad (6)$$

where  $I_1$  and  $K_1$  are modified Bessel functions of first order and  $D$  is an integration constant. Constants  $A$  and  $B$  are evaluated from equation

$$V_1 = A I_0 (\theta / n) + B K_0 (\theta / n) \quad (7)$$

$$V_2 = A I_0 (\theta) + B K_0 (\theta) \quad (8)$$

Define

$$\Delta_1 = \begin{vmatrix} I_O(\theta/n) & I_O(\theta) \\ K_O(\theta/n) & K_O(\theta) \end{vmatrix}$$

then

$$A = [V_1 K_O(\theta) - V_2 K_O(\theta/n)] / \Delta_1 \quad (9)$$

$$B = [V_1 I_O(\theta) - V_2 I_O(\theta/n)] / \Delta_1 \quad (10)$$

and constant D equals zero by integrating Eq.(2) to yield

$$\begin{aligned} V_2 - V_1 = \int I dr = & [A I_O(\theta) - I_1(\theta/n)] + [B K_O(\theta) - K_O(\theta/n)] \\ & - DR' \log n \end{aligned} \quad (11)$$

and substituting Eqs. (9) and (10) in Eq. (11).

Substituting Eqs. (9) and (10) in Eqs. (5) and (6), Eqs. (5) and (6) reduce to

$$I_1 = -V_1 \Delta_2 / \Delta_1 R' - V_2 \Delta_3 / \Delta_1 R' \quad (12)$$

$$I_2 = -V_1 \Delta_4 / \Delta_1 R' - V_2 \Delta_5 / \Delta_1 R' \quad (13)$$

where

$$\Delta_2 = \theta \begin{vmatrix} K_O(\theta) & K'_O(\theta/n) \\ I_O(\theta) & I'_O(\theta/n) \end{vmatrix}$$

$$\Delta_3 = \theta \begin{vmatrix} I_O(\theta/n) & I'_O(\theta/n) \\ K_O(\theta/n) & K'_O(\theta/n) \end{vmatrix}$$

$$\Delta_4 = \theta \begin{vmatrix} K_O(\theta) & K'_O(\theta) \\ I_O(\theta) & I'_O(\theta) \end{vmatrix}$$

$$\Delta_5 = \theta \begin{vmatrix} I_O(\theta/n) & I'_O(\theta) \\ K_O(\theta/n) & K'_O(\theta) \end{vmatrix}$$

From Eqs. (12) and (13)

$$(y) = R'$$

$$(h) = -\frac{\Delta_2}{\Delta_1}$$

$$(b) = \frac{\Delta_3}{\Delta_1}$$

$$(a) = \frac{\Delta_4}{\Delta_1}$$

$$(g) = \frac{\Delta_5}{\Delta_1}$$

using the identity

$$I_1(x)K_O(x) + K_I(x)I_O(x) = 1$$

yields

$$\Delta_3 = -\Delta_4 = -1$$

$$\text{and } (a) = (b) = -\frac{1}{\Delta_1}$$

hence

$$(z) = -\frac{[1 + \Delta_2 \Delta_5]}{\Delta_1^2 R'}$$

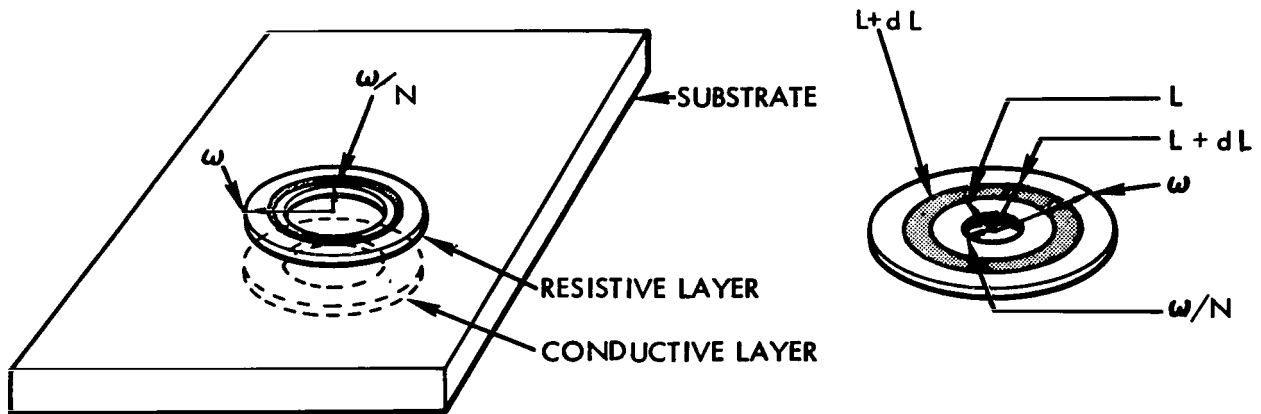


Figure 8.- Incremental model of Circular D-P network

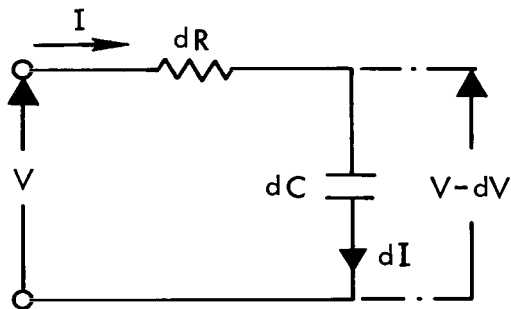


Figure 9.- A schematic of Figure 8

### Asymptotes for Tapered Networks

The deviations for a circular network, summarized in column 1 of Table XII, reveal that  $a = b$  and  $g \neq h$ , hence, the network is asymmetric and passive, as expected.

Proceedings as in the case of the rectangular network asymptotes at high frequency and low frequency can be found for amplitude and phase. Proceedings as before, break-points can be calculated for specific values of  $n$ , or if warranted, universal curves for the break-point frequency as a function of  $n$  can be plotted. Typical values are given in Table XII. The asymptotes are plotted in Figures 10 and 11.

### RELATED APPLICATIONS

To illustrate the usefulness of asymptotic techniques to distributed-parameter circuits, the following examples are considered:

TABLE XII.- DEVIATIONS FOR D-P NETWORK WITH CIRCULAR GEOMETRY

		Magnitude Asymptotes		Break-Point		Phase Asymptotes	
Deviations	$f(x)$	$ f(x) _{x \rightarrow 0}$	$ f(x) _{x \rightarrow \infty}$	arbitrary $n$	$n=1.5$	$\angle f(x) _{x \rightarrow 0}$	$\angle f(x) _{x \rightarrow \infty}$
(y)	$-R'$	$-R'$	$-R'$			0	0
(h)	$-\Delta_2/\Delta_1$	$1/\log n$	$(x)^{1/2}$	$n^2/(\log n)^2$	13.7	0	$+\pi/4$
(g)	$\Delta_5/\Delta_1$	$1/\log n$	$(x)^{1/2}$	$\frac{1}{(\log n)^2}$	6.1	0	$+\pi/4$
(a)	$-\Delta_4/\Delta_1$	$1/\log n$	$(n/x)^{1/2} e^{(x/2)^{1/2}(1-1/n)}$		.3	0	$\frac{3\pi}{4} + (x/2)^{1/2} \left(1 - \frac{1}{n}\right)$
(b)	$\Delta_3/\Delta_1$	$1/\log n$	$(n/x)^{1/2} e^{(x/2)^{1/2}(1-1/n)}$		.3	0	$\frac{3\pi}{4} + (x/2)^{1/2} \left(1 - \frac{1}{n}\right)$
(z)	$-(1 + \Delta_2 \Delta_5)/\Delta_1^2 R'$	$xK/(\log n)^2 R'$	$x/nR'$			$\pi/2$	$\pi/2$

$R' = R/\log n$

$K = 308/n^2 + (1 - 1/n^2) .5 \log n$



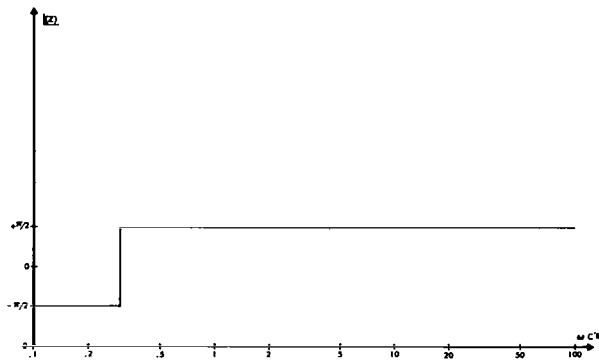
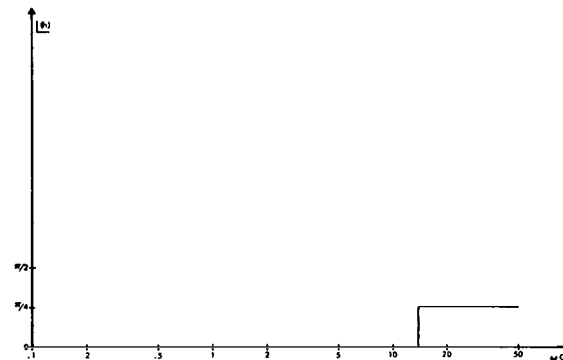
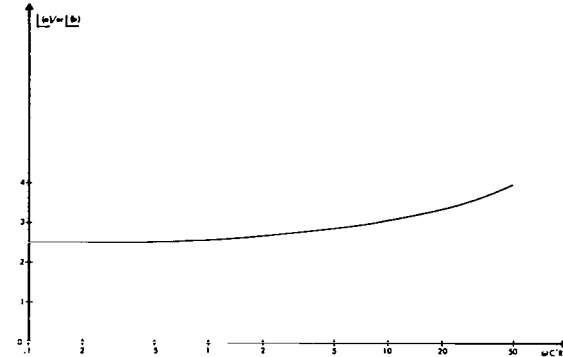
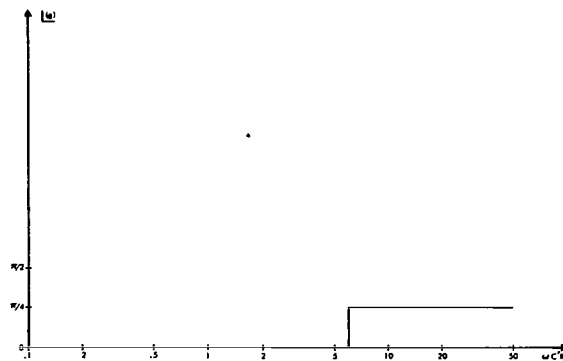
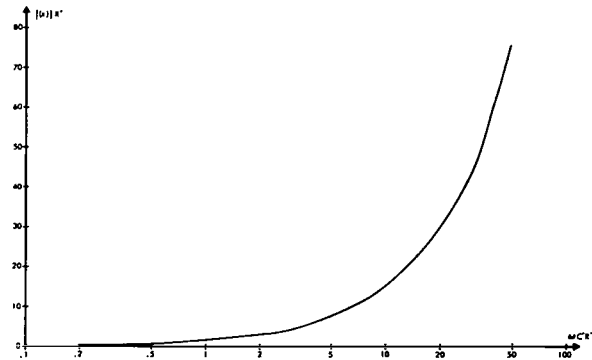
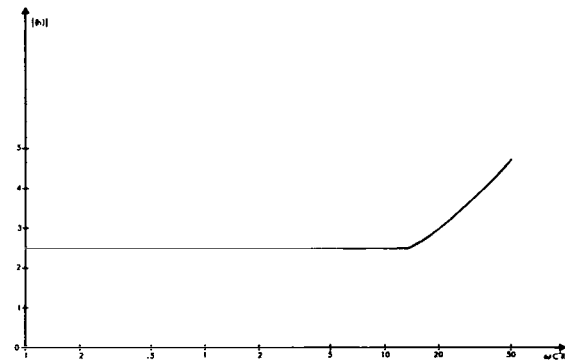
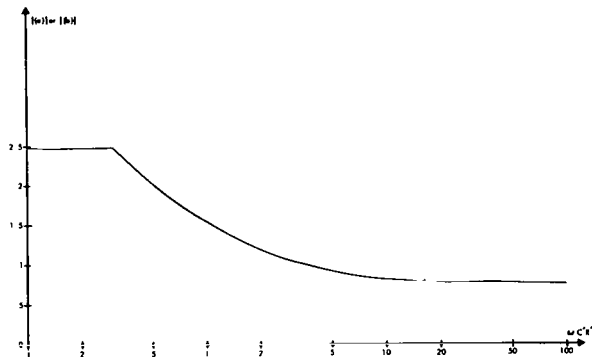
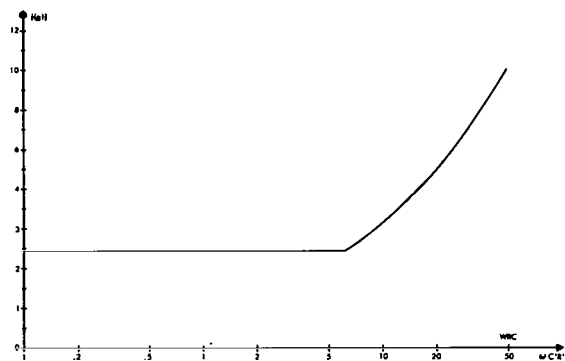


Figure 10.- Magnitude asymptotes of deviations of circular network

Figure 11.- Phase asymptotes of deviations of circular network

(1) determination of first null frequencies of four types of null filters shown in Table XIII.

(2) evaluation of transient response of a simple d-p network (Figure 2).

Null Frequency.- The first null frequency of each of four filters is determined using approximate methods, instead of solving transcendental Eq. (1). The filter (a) in Table XII is studied in detail; results for other networks are tabulated.

An open circuit voltage-to-voltage transfer function for the network (a) is given by

$$g_{21} = v_2/v_1 \Big|_{I_2 = 0} = \left[ (a) + R_1(z) \right] / \left[ (g) + R_1(z) \right]$$

$$= \frac{(\theta/\sinh \theta) + R_1(\theta^2/R)}{[(\theta \coth \theta) + (\theta^2/R)R_1]}$$

where

$$\begin{aligned} C &= \lambda c_o \\ R &= \lambda r_o \\ c_o &= \text{capacitance per unit length of the d-p network} \\ r_o &= \text{resistance per unit length of the network} \\ \lambda &= \text{length of the film.} \end{aligned}$$

No generality is lost if one assumed that  $R = C = 1$ . Null frequencies are obtained by setting both the imaginary part and the real part of  $g_{21}$  simultaneously equal to zero

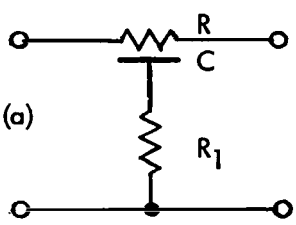
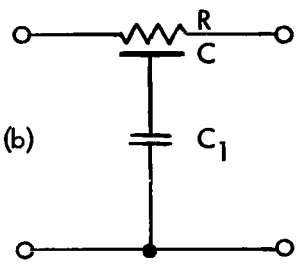
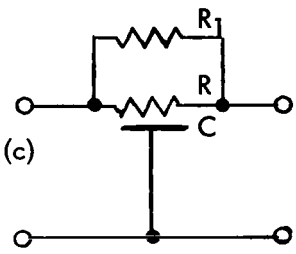
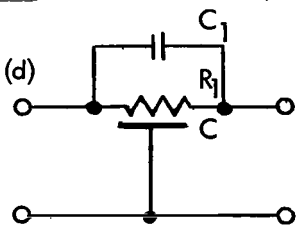
$$\text{Re}(g_{21}) \quad \text{and} \quad \text{Im}(g_{21}) = 0$$

These expressions yield the null frequencies

$$\tan(\omega/2)^{1/2} = -\tanh(\omega/2)^{1/2}$$

Kaufman (ref. 12) has proposed a graphical solution of this equation.

TABLE XIII.- NULL FILTERS AND FIRST NULL FREQUENCIES

Filter Configuration	$g_{21}$	Null frequency (Radians/sec)	$R_1$ or $C_1$
(a) 	$\frac{\theta/\sinh \theta + R_1 (\theta^2/R)}{\theta \coth \theta + (\theta^2/R)R_1}$	$10.95/(RC)$	$R/17.143$
(b) 	$\frac{j\omega C_1 \theta + \theta^2 \sinh \theta/R}{j\omega C_1 \theta \cosh \theta + \theta^2 \sinh \theta/R}$	$28.98/(RC)$	$6C$
(c) 	$\frac{R_1 \theta + R \sinh \theta}{R_1 \theta \cosh \theta + R \sinh \theta}$	$28.98/(RC)$	$6R$
(d) 	$\frac{\theta/\sinh \theta + j\omega RC_1}{\theta \coth \theta + j\omega RC_1}$	$10.95/(RC)$	$C/17.143$

Null frequencies, given by points of intersection of  $\tan(\omega/2)^{1/2}$  and  $-\tanh(\omega/2)^{1/2}$  curves, define  $R_1$  for which a null is obtained from the two equations for  $g_{21}$ :

$$(2)^{1/2} \sin(\omega/2)^{1/2} \cosh(\omega/2)^{1/2} = 1/R_1$$

By graphical solution, Kaufman (ref. 14) has found various null frequencies and the corresponding values of  $R_1$ . Our purpose is to find the null frequency by approximation technique i.e., by approximating the transcendental function to a rational one. Only the first null frequency and corresponding value of  $R_1$  are considered.

The null is defined by

$$\theta \sinh \theta = -R/R_1$$

By Taylor series expansion

$$\theta(\theta + \theta^3/3! + \theta^5/5! + \theta^7/7! + \dots) = -R/R_1$$

or

$$i[\omega CR - (\omega CR)^3/5!] - [(\omega CR)^2/3! - (\omega CR)^4/7!] = -R/R_1$$

Thus, the first null frequency is given by

$$\omega_0 CR = 10.95 \quad \text{and of course} \quad \omega_0 CR = 0$$

and

$$R/R_1 = (\omega_0 CR)^2/3! - (\omega_0 CR)^4/7! = 17.132$$

It is interesting to compare these values with those obtained by Kaufman (ref. 14) through the graphical approach. His results are compared in Table XIV. Thus, this approximation yields the desired results with an accuracy within 2.5 percent. Null frequencies and the corresponding lumped elements for other filters are shown in Table XIII.

TABLE XIV.- COMPARISON OF RESULTS

	Graphical	Asymptotic	% Discrepancy
$\omega_0 CR$	11.18	10.95	2.05
$R/R_1$	17.786	17.143	3.61

Transient Response.- Transient response of a simple RC distributed network shown in Figure 2 is considered. An open circuit voltage-to-voltage transfer function is given

$$v_2(\theta)/v_1(\theta) \Big|_{I_2} = g_{21}(\theta) = 1/\cosh \theta$$

or

$$v_2(s)/v_1(s) = 1/\cosh(sCR)^{1/2}$$

For a step input  $E_1(s) = 1/s$ , so that

$$\begin{aligned} v_2(s) &= (1/s) \cosh(sCR)^{1/2} \\ &\approx 1/s \left[ 1 - sCR/2 + 5(sCR)^2/24 - 61(sCR)^3/720 \right] \\ &\approx 1/s - (CR/2) \left[ 1 - 5sCR/12 + 61s^2C^2R^2/360 \right] \\ &\approx 1/s - (CR/2) \left( \frac{1 + 5sCR}{12} \right)^{-1} + \text{terms in } s^2 \end{aligned}$$

Hence, for  $t \gg 1$ ,

$$v_2(t) = 1 - (6/5) \exp(-12/5CR)t$$

At  $t/CR = 2$

$$v_2(t) \approx .9897$$

This checks satisfactorily with the experimental value .99 given by Kaufman and Garrett (ref. 13). A comparison of techniques for the transient response is presented by Chen and Bauer (ref. 17) and others (refs. 18, 19).

## CONCLUSION

The customary presentation of network functions by six conventional sets of circuit parameters,  $a$ ,  $b$ ,  $g$ ,  $h$ ,  $y$  and  $z$  has been replaced in this investigation by the introduction of the concept of deviations. To achieve economy and efficiency in evaluating correction curves for a wide variety of distributed-parameter network functions, it was necessary to establish a set which is sufficient and necessary to derive all possible network functions. For a  $\overline{RC}$  two-port it has been necessary to select only seven functions referred to as deviations. Evaluation of network functions (Tables VII, VIII, and XIII) is accomplished by a systematic procedure. Rules for deviations are stated and tables presented for reference to accomplish operations of substantial complexity.

Asymptotic techniques for frequency-domain analysis used extensively in lumped parameter networks are extended to distributed-parameter networks by expansion of deviations at high and low frequencies and by establishing "break-points". Delineating the range of validity of each asymptotic approximation, frequency asymptotes are to be supplemented with error curves to provide the engineer with more accurate data for design purposes, in case the asymptotic approximations fail to meet his requirements.

Approximations of transcendental functions by rational functions give a maximum error of 10 percent as a function of frequency. With the use of error-correcting curves an accuracy better than 1 percent may be achievable.

Distributed parameter network functions are transcendental functions with an infinite number of roots and conversion from frequency-domain to time-domain involves evaluation of residues at an infinite number of poles. The conventional technique of conversion can be replaced by a sufficiently accurate approximation obtained by asymptotic expansion of transcendental functions. The asymptotic expansion of transcendental network functions yield reasonable accuracy in many cases if applied to time-domain analysis.

## REFERENCES

1. Bower, J. L.; and Schultheiss, P. M.: Introduction to the Design of Servomechanism. John Wiley and Sons, 1968.
2. James, H. M.; Nichols, N. B.; and Phillips, R. S.: Theory of Servomechanism. McGraw Hill, 1947.
3. Thaler, G. K.; and Brown, G. S.: Analysis and Design of Feedback Control Systems. McGraw Hill, 1960.

4. Bode, H. W.: Network Analysis and Feedback Amplifier Design. D. Van Nostrand Co., 1945.
5. Nisbet, T. R.; and Happ, W. W.: Calculus of Deviations Applied to Transistor Circuit and Network Analysis. J. Brit. Instit. Radio Eng., Vol. 21, 1961, pp. 437-450.
6. Krajewski, A. S.; and Happ, W. W.: Deviacje - Nowa Metoda Prezekosztatcew Parametrow Elektrycznych, Rozprawy Elektrotechniczne, Tom XII, 4, pp. 675-706, Warsaw, Sept. 1966.
7. Shekel, J.: Matrix Representation of Transistor Circuits, Proc. Inst. Radio Eng., Vol. 40, Nov. 1952.
8. Happ, W. W.: Deviations - A Twenty-Year Old Calculus for Multitechnical Networks. Proc. Midwest Symposium on Circuit Theory, Section IX.6, Austin, Texas, April 1969.
9. Happ, W. W.; and Gupta, S. C.: Time Domain Analysis and Measurement Techniques for RC Structures, J. Appl. Phys., Vol. 40, No. 1, pp. 109-117, Feb. 1969.
10. Bhattacharyya, B. B.; Stein, R. A.; and Swamy, M. N. S.: On Multiwire Non-Uniform Transmission Lines. Proc. Midwest Symposium on Circuit Theory, Section II.1, Austin, Texas, April 1969.
11. Wyndrum, R. W.: Distributed RC Network Synthesis - What Important Challenges Remain. Proc. Allerton Conference on Circuit and System Theory, p. 286-295, IEE Cat. No. 68 C44 CT, Oct. 1968.
12. Heizer, K. W.: Distributed RC Network with Rational Transfer Function. J.R.E. Transactions on Circuit Theory, Dec. 1962.
13. Kaufman, W. M.; and Garrett, S. J.: Tapered Distributed Filters. IRE Transactions on Circuit Theory, Vol. CT-9 No. 4, pp. 329-336, Dec. 1962.
14. Kaufman, W. M.: Theory of a Monolithic Null Devices and Same Novel Circuits. Proc. IRE, Vol. 48, pp. 1540-1545, Sept. 1960.
15. Edson, W. A.: Tapered Distributed RC Lines for Phase-shift Oscillators. Proc. IRE, June, 1961.
16. Castro, P. S.; and Happ, W. W.: Distributed Parameter Circuit Design Techniques. Proc. National Electronics Conference, Palo Alto, California, Vol. 17, pp. 47-70, 1961.
17. Chen, T. H.; and Bauer, R. F.: On the Transient Response of Distributed RC Lines. Proc. Midwest Symposium on Circuit Theory, Section XIII, Austin, Texas, April 1969.

18. Lindholm, F. A.; and Happ, W. W.: Transient Analysis of Thin-Film Structures. J. Brit. Instit. Radio Eng., Vol. 26, pp. 421-435, 1963. Also Microelectronics and Reliability, Vol. 3, pp. 209-226, 1964.
19. Happ, W. W.; and Standhammer, J.: Critical Excitation Functions of Distributed Parameter Devices. IEEE Trans. Vol. ED-13, No. 3, pp. 347-361, March 1968.
20. Happ, W. W.: Asymptotes for Frequency-Response of Distributed-Parameter Components. Proc. Midwest Symposium on Circuit Theory, pp. 89-94, Notre Dame, Indiana, May 1968.



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